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Polyhedral (and Approximation) Aspects of FVS and PFDS

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Problem Definitions and Motivating Questions

Input: Undirected Graph $G = (V, E)$ with vertex weights $w : V \to \mathbb{R}_+$.

Feedback Vertex Set (FVS) : $\min_{F \subseteq V} \{w(F) : G - F \text{ has no cycles}\}$.

- Hitting set LP: $\min\{w^T x : x \in \mathcal{P}_{\text{cycle-cover}}(G)\}$, where

$$\mathcal{P}_{\text{cycle-cover}}(G) := \left\{ x \in \mathbb{R}^V_{\geq 0} : \sum_{u \in U} x_u \geq 1 \quad \forall U \subseteq V : G[U] \text{ is a cycle} \right\}.$$

- FVS is NP-hard and $(2 - \epsilon)$-inapproximable under UGC.
- FVS admits combinatorial 2-approx algorithm (local-ratio).
- FVS admits LP-based 2-approx algorithm (primal-dual on $\min\{w^T x : x \in \mathcal{P}_{\text{SD}}(G)\}$).

$$\mathcal{P}_{\text{SD}}(G) := \left\{ x \in \mathbb{R}^V_{\geq 0} : \sum_{u \in S} (d_S(u) - 1) x_u \geq |E[S]| - |S| + 1 \quad \forall S \subseteq V : E[S] \neq \emptyset \right\}$$

Remark. Constructing a polytime separation oracle for $\mathcal{P}_{\text{SD}}(G)$ constraints is an open problem.

Question 1.1. Does there exist an ILP formulation for FVS whose LP-relaxation can be solved in polynomial time and has an integrality gap at most 2?

Conjecture 1.2 (open). Let $x \in \mathcal{P}_{\text{SD}}(G)$ be an extreme point. Then, there exists $u \in V$ such that $x_u \geq 1/2$.

Subset Feedback Vertex Set (SFVS) : $\min_{F \subseteq V} \{w(F) : G - F \text{ has no cycles containing terminals } T \subseteq V\}$.

- Terminal set $T = V$ gives back FVS.
- SFVS is NP-hard and $(2 - \epsilon)$-inapproximable under UGC.
- SFVS admits a combinatorial 8-approx algorithm (relaxed multicommodity flows).
- SFVS admits LP-based 13-approx algorithm (Chekuri-Madan *poly-sized* labeling LP).

Remark. Determining the exact integrality gap of Chekuri-Madan LP for SFVS is an open problem.

Question 2.1. Is the integrality gap of Chekuri-Madan LP at most 2 for FVS?

PseudoForest Deletion Set (PFDS) : $\min_{P \subseteq V} \{w(P) : \underbrace{\text{Connected components of } G - P \text{ have } \leq 1 \text{ cycle}}_{G - P \text{ is a pseudoforest}}\}$.

- Hitting set LP: $\min\{w^T x : x \in \mathcal{P}_{\text{2PT-cover}}(G)\}$, where

$$\mathcal{P}_{\text{2PT-cover}}(G) := \left\{ x \in \mathbb{R}^V_{\geq 0} : \sum_{u \in U} x_u \geq 1 \quad \forall U \subseteq V : G[U] \text{ contains} \geq 2 \text{ cycles} \right\}$$

- PFDS is NP-hard and $(2 - \epsilon)$-inapproximable under UGC.
- PFDS has a combinatorial 2-approx algorithm (local-ratio).
- PFDS has a LP-based 2-apx algorithm (primal-dual on $\min\{w^T x : x \in \mathcal{P}_{\text{WD}}(G) \cap \mathcal{P}_{\text{2PT-cover}}(G)\}$)

$$\mathcal{P}_{\text{WD}}(G) := \left\{ x \in \mathbb{R}^V_{\geq 0} : \sum_{u \in S} (d_S(u) - 1) x_u \geq |E[S]| - |S| \quad \forall S \subseteq V \right\}$$

Remark. Constructing a polytime separation oracle for $\mathcal{P}_{\text{WD}}(G)$ constraints is an open problem

Question 3.1. Does there exist an ILP formulation for PFDS whose LP-relaxation can be solved in polynomial time and has an integrality gap at most 2?

Question 3.2 (Motivated by Conjecture 1.2). Does there exists a constant $\alpha > 0$ such that for every extreme point $x \in \mathcal{P}_{\text{WD}}(G)$, there exists $u \in V$ with $x_u \geq \alpha$.

Main Results.

New ILP Formulations.

$$\mathcal{P}_{\text{orient}}(G) := \left\{ (x, y) : \begin{array}{ll} x_u + x_v + y_{e,u} + y_{e,v} \geq 1 & \forall e = \{u, v\} \in E \\ x_u + \sum_{e \in \delta(u)} y_{e,u} \leq 1 & \forall u \in V \\ x, y \geq 0 & \end{array} \right\}$$

$$\mathcal{Q}_{\text{orient}}(G) := \left\{ x \in \mathbb{R}^V : (x, y) \in \mathcal{P}_{\text{orient}}(G) \right\}$$

Remark 1. *For every graph G, we have that $\mathcal{Q}_{\text{orient}}(G) \subseteq \mathcal{P}_{\text{WD}}(G)$; moreover, there exist graphs for which inclusion is strict.*

Remark 2. *The definition of $\mathcal{P}_{\text{orient}}(G)$ is motivated by the dual of Charikar's LP for Densest Subgraph.*

The following is an ILP formulation for PFDS:

$$\min\{w^T x : x \in \mathcal{Q}_{\text{orient}}(G) \cap \mathbb{Z}^V\}.$$

The following are two ILP formulations for FVS:

$$\min\{w^T x : x \in \mathcal{P}_{\text{WD}}(G) \cap \mathcal{P}_{\text{cycle-cover}}(G) \cap \mathbb{Z}^V\}.$$
$$\min\{w^T x : x \in \mathcal{Q}_{\text{orient}}(G) \cap \mathcal{P}_{\text{cycle-cover}}(G) \cap \mathbb{Z}^V\}.$$

Integrality Gap Results (Polytime Solvable Formulations).

Theorem 1. *The integrality gap of the following LP is at most 2 for PFDS:*

$$\min\{c^T x : x \in \mathcal{Q}_{\text{orient}}(G) \cap \mathcal{P}_{\text{2PT-cover}}(G)\}.$$

Theorem 2. *The integrality gap of the following LP is at most 2 for FVS:*

$$\min\{w^T x : x \in \mathcal{Q}_{\text{orient}}(G) \cap \mathcal{P}_{\text{cycle-cover}}(G)\}.$$

Theorem 3. *There exists a polynomial-sized ILP formulation for FVS whose LP-relaxation has integrality gap at most 2. In particular,*

1. *the integrality gap of $\min\{c^T x : x \in \mathcal{Q}_{orient}(G) \cap \mathcal{P}_{cycle-cover}(G)\}$ is at most 2,*
2. *the integrality gap for the Chekuri-Madan LP for FVS is at most 2,*
3. *(informal) there exists an orientation-based LP without cycle-cover constraints with integrality gap at most 2.*

Extreme Point Results.

Theorem 4. *Let G be graph that is not a pseudoforest and let $x \in \mathcal{P}_{\text{WD}}(G)$ be an extreme point. Then, there exists a vertex $u \in V$ such that $x_u \geq 1/3$. Furthermore, there exists a graph G for which the inequality is tight.*

Remark. To prove Theorem 4, we use *Conditional Uncrossing*, a new technique described in the next section of the poster.

Theorem 5. *Let G be graph that is not a pseudoforest and let $x \in \mathcal{P}_{\text{orient}}(G)$ be a minimal extreme point. Then, there exists a vertex $u \in V$ such that $x_u \geq 1/3$. Furthermore, there exists a graph G for which the inequality is tight.*

By using Theorem 4 and Theorem 5 with the *iterated rounding* framework, we immediately get the following two corollaries.

Corollary 5.1. *The integrality gap of the following LP is at most 3 for PFDS:*

$$\min\{w^T x : x \in \mathcal{Q}_{\text{orient}}(G)\}.$$

Corollary 4.1. *The integrality gap of the following LP is at most 3 for PFDS:*

$$\min\{w^T x : x \in \mathcal{P}_{\text{WD}}(G)\}.$$

Conditional Uncrossing: A New Technique.

Extreme point properties of polyhedra (similar to Conjecture 1.2) can be shown when the constraints have underlying submodularity/supermodularity structure. This allows *tight* constraints at extreme points to be *uncrossed* to get well-structured families (chain, laminar, cross-free, etc.) of linearly independent tight constraints. However, recall that

$$\mathcal{P}_{\text{WD}}(G) := \left\{ x \in \mathbb{R}^V_{\geq 0} : \underbrace{|E[S]| - |S| - \sum_{u \in S} (d_S(u) - 1) x_u}_{=: f_x(S)} \leq 0 \quad \forall S \subseteq V \right\}$$
$$= \left\{ x \in \mathbb{R}^V_{\geq 0} : \underbrace{\sum_{uv \in E[S]} (1 - x_u - x_v)}_{=: p_x(S)} - \underbrace{\sum_{u \in S} (1 - x_u)}_{=: q_x(S)} \leq 0 \quad \forall S \subseteq V \right\}$$

Note that f_x is not supermodular in general. Nevertheless, we prove Theorem 4 with the following crucial observation.

Observation (Conditional Supermodularity). Let $x \in \mathcal{P}_{\text{WD}}$ such that $x_u \leq \frac{1}{2}$ for each $u \in V$. Then, the function f_x is supermodular (since p_x is supermodular and q_x is modular).

Strategy to Prove Theorem 4

1. Consider an arbitrary extreme point $x \in \mathcal{P}_{\text{WD}}(G)$.
2. Assume by way of contradiction that $x_u < 1/3$ for each $u \in V$.
3. Consider the submatrix A of constraints of $\mathcal{P}_{\text{WD}}(G)$ that are equations at the vector x.
4. Show that $\texttt{row-rank}(A) < |V| = \texttt{dimension}\,(\mathcal{P}_{\text{WD}}(G)) = \texttt{column-rank}(A)$, a contradiction.

Additional Details I: Conditional Uncrossing Properties

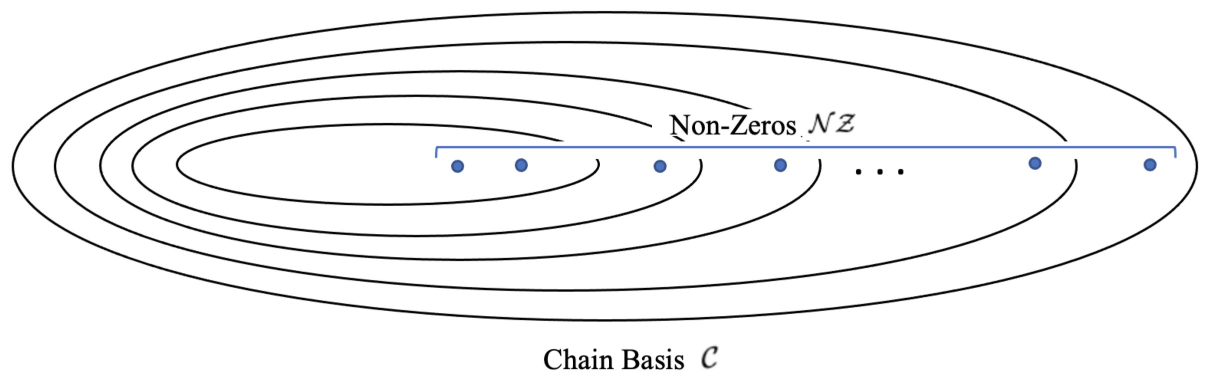
Let x be an extreme point of $\mathcal{P}_{\text{WD}}(G)$ such that $x_u < \frac{1}{2}$ for each $u \in V$ and the family of *tight sets* for x be $\mathcal{T} := \{S \subseteq V : f_x(S) = 0\}$. Let $A, B \in \mathcal{T}$. Then,

- $A \cap B \neq \emptyset$,
- $A \cap B, A \cup B \in \mathcal{T}$,
- $\text{row}(A) + \text{row}(B) = \text{row}(A \cap B) + \text{row}(A \cup B)$, i.e. constraint-matrix vectors of $\mathcal{P}_{\text{WD}}(G)$ for tight sets uncross.

Additional Details II: Conditional Basis

Let $x \in \mathcal{P}_{\text{WD}}(G)$ be an *extreme point* such that $x_u < \frac{1}{3}$ for each $u \in V$. Let $\mathcal{T} := \{S : f_x(S) = 0\}$ and $\mathcal{NZ} := \{u : x_u \neq 0\}$. Then, there exists a family $\mathcal{C} \subseteq \mathcal{T}$ such that

- The family $\mathcal{C}$ is a *chain family* such that the vectors $\texttt{Rows}(\mathcal{C})$ are linearly independent,
- For each $A, B \in \mathcal{C}$ such that $A \subset B$, there exists a vertex $u \in (B - A) \cap \mathcal{NZ}$.
- For each $A \in \mathcal{C}$, there exist distinct vertices $u, v \in A \cap \mathcal{NZ}$,
- $\texttt{rank}(\texttt{Rows}(\mathcal{C})) = |\mathcal{NZ}|$.



First three properties contradict the last property.

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